Journal of Engineering Mathematics, Vol. 9, No. 3, July 1975
Noordhoff International Publishing–Leyden
Printed in The Netherlands

Plastic-elastic torsion, optimal stopping and free boundaries

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(Received November 21, 1974)

SUMMARY

It is shown that the mathematical formulation of the plastic-elastic torsion of a cylindrical bar with cross section S and that of optimal stopping of Brownian motion on S with its boundary absorbing, with cost of motion per unit of time a constant and with stopping costs described by a surface of constant slope on the boundary of S both lead to the same partial differential equation with the same free boundary conditions. The well-known membrane-sandhill analogy of Nádai is here then also a very useful interpretation of the optimal stopping problem.

Also the discrete models for both problems are considered, their analogy is shown; and some numerical procedures, in particular an L. P. formulation, are pointed out.

1. Introduction

The mathematical formulation of the plastic-elastic torsion phenomenon of a cylindrical bar leads to a partial differential equation with unknown boundaries. In many books on plasticity the membrane-sandhill analogy is used to illustrate the solution of the just mentioned partial differential equation and to elucidate the role of the free boundary.

The central problem in optimal stopping theory of processes evolving in time is the determination of an optimal stopping time such that an associated cost function reaches a minimum. The solution of such problems frequently amounts to finding the optimal set of stopping states, i.e. optimal stopping is realized if upon entrance of the process into such a state the process is stopped. The determination of such optimal stopping sets is actually a problem with unknown boundaries.

In the present study it will be shown that optimal stopping of Brownian motion on the cross section of the bar with its boundary absorbing states, with stopping cost function a surface congruent with that of the maximal sandhill on the cross section (see Section 4) and with constant cost for motion per unit of time leads to the same mathematical problem as that of plastic-elastic torsion. Similarly, the finite difference approximations of the plastic-elastic torsion and of the optimal stopping are identical.

The finite difference approximations are usually applied to obtain numerical solutions. Since several numerical procedures for discrete optimal stopping problems have been developed, these techniques may be of use also for the numerical evaluation of plastic-elastic torsion problems.

The membrane-sandhill analogy of Nádai (see Section 4) is often used to illustrate the plastic-elastic torsion phenomenon. From the discussions in the next section it will be seen that this analogy is also useful to obtain an insight in the optimal stopping of Brownian motion; although here, however, a very special cost function is considered (that congruent with the sandhill) it is readily seen that this is not essential, i.e. the analogy is easily extended to other type of cost functions.

2. The discrete mechanical model

Consider a hammock with square meshes (side $= c$) stretched over a rectangular window with sides a and b. Over this rectangular is a roof of which the four faces have the same slope α with the plane of the rectangle, see figure.

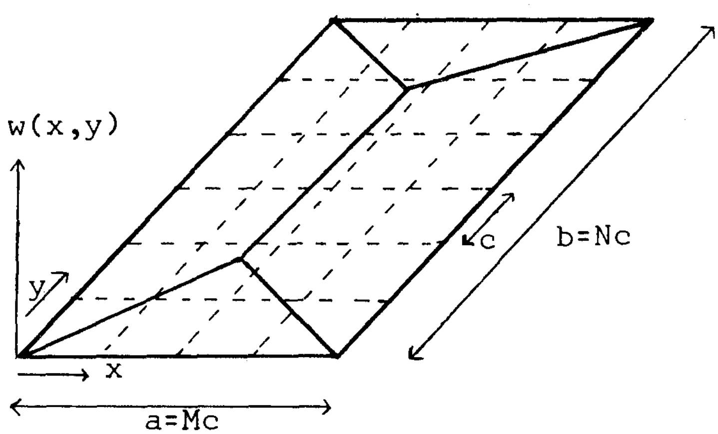

Figure 1.

At the inner nodes of the hammock forces of equal magnitude p are applied, positive in the upward direction. The strings of the hammock are not fixed to each other at the internal nodes and are all stretched with the same force s. For p sufficiently large all internal nodes will be pressed against the roof.

For given p and s it is required to determine the equilibrium position of the hammock.

Denote by $w(x, y)$ the surface of the roof and by $\Phi(x, y)$ the equilibrium state of the node initially at point (x, y).

It is readily seen that for small displacements of the nodes the conditions of equilibrium for a node are

$$\begin{aligned}
&\frac{pc}{4s} + \Delta_c \Phi = 0\,, \quad \text{if} \quad \Phi < w\,, \quad \text{for } (x, y) \in U_i\,,\\
&\frac{pc}{4s} + \Delta_c \Phi \geqq 0\,, \quad \text{if} \quad \Phi = w\,, \quad \text{for } (x, y) \in U_i\,,\\
&\Phi = 0\,, \qquad \text{for } (x, y) \in U_b\,,
\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
&U_i \overset{\text{def}}{=} \{(x, y)\colon\ x = mc,\ y = nc\,;\ m = 1, \ldots, M-1\,;\ n = 1, \ldots, N-1\}\,,\\
&U \overset{\text{def}}{=} \{(x, y)\colon\ x = mc,\ y = nc\,;\ m = 0, \ldots, M\,;\quad n = 0, \ldots, N\}\,,\\
&U_b \overset{\text{def}}{=} U - U_i\,,
\end{aligned} \tag{2.2}$$

$$\Delta_c \Phi(x, y) \overset{\text{def}}{=} \tfrac{1}{4}\{\Phi(x-c, y) + \Phi(x+c, y) + \Phi(x, y-c) + \Phi(x, y+c)\} - \Phi(x, y)\,. \tag{2.3}$$

The conditions (2.1) are obviously equivalent with

$$\begin{aligned}
&\Phi = \min\left\{w, \frac{pc}{4s} + \Delta_c \Phi + \Phi\right\}, \qquad (x, y) \in U_i\,,\\
&\Phi = 0\,, \qquad (x, y) \in U_b\,.
\end{aligned} \tag{2.4}$$

3. Optimal stopping, discrete model

Let $\{z_n, n=0, 1, \ldots\}$ denote the position after n steps of a point subjected to a symmetric random walk on U (*cf.* (2.2)) with the states of U_b being absorbing states and with transition probabilities from (x, y) to neighbouring states equal to $\frac{1}{4}$ for $(x, y) \in U_i$. Let $\boldsymbol{n}$ be a stopping time* for the process $\{z_n, n=0, 1, \ldots\}$, i.e. for every n the event $\{\boldsymbol{n}=n\}$ depends only on z_m, $m=0, \ldots, n$. Since the entrance time into the boundary is finite with probability one, and has a finite expectation, all stopping times are finite and have finite expectation. Denote by ζ_n the stopping process belonging to the stopping time $\boldsymbol{n}$, i.e.

$$\begin{aligned} \zeta_n &= z_n \quad \text{for} \quad n < \boldsymbol{n}, \\ &= z_{\boldsymbol{n}} \quad \text{for} \quad n \geqq \boldsymbol{n}. \end{aligned} \tag{3.1}$$

With every stopping process a cost function is associated: As long as $n < \boldsymbol{n}$ a cost $k > 0$ has to be paid per transition to a neighbouring point, while stopping of the process at a point (ξ, η) involves a cost $w(\xi, \eta)$, with

$$\begin{aligned} w(\xi, \eta) &\geqq 0 \quad \text{for} \quad (\xi, \eta) \in U_i, \\ &= 0 \quad \text{for} \quad (\xi, \eta) \in U_b. \end{aligned} \tag{3.2}$$

Hence for a given stopping time $\boldsymbol{n}$ the expectation of the total costs of the with $\boldsymbol{n}$ associated stopping process (*cf.* (3.1)), when started in $\zeta_0 = (x, y)$, is given by

$$\phi(x, y) \overset{\text{def}}{=} E\{k\boldsymbol{n} + w(z_{\boldsymbol{n}}) | z_0 = (x, y)\}, \qquad (x, y) \in U, \tag{3.3}$$

with $z_{\boldsymbol{n}} = (\xi, \eta)$ if the process stops at (ξ, η).

Define

$$\Phi(x, y) \overset{\text{def}}{=} \inf_{\boldsymbol{n}} \phi(x, y), \qquad (x, y) \in U, \tag{3.4}$$

i.e. Φ is the infimum of the expected costs over all possible stopping processes.

A stopping time $\boldsymbol{\sigma}_0$ is called optimal if for the associated stopping process the expected costs are equal to Φ for all $(x, y) \in U$, i.e. $\boldsymbol{\sigma}_0$ describes our optimal policy if we are compelled to play this game. Starting the process at (x, y) and acting optimally, if an optimal policy exists, we decide to stop immediately i.e. pay $w(x, y)$ if this is less or equal to the sum of the cost k of making one step and the average expected cost when starting in a neighbouring point of (x, y), hence (*cf.* (2.3))

$$\begin{aligned} \Phi &= \min\{w, k + \Delta_c \Phi + \Phi\} \quad &\text{for} \quad (x, y) \in U_i, \\ \Phi &= 0 \quad &\text{for} \quad (x, y) \in U_b, \end{aligned} \tag{3.5}$$

where the second relation follows from (3.2).

Taking for the stopping cost function w in (3.5) the "roof" function w of the preceding section, it is seen that the determination of the optimal stopping process and that of the equilibrium state of the hammock are identical. If $k = pc/4s$ then (x, y) is a stopping state if here $w = \Phi$, i.e. the node of the hammock at (x, y) touches the roof.

For optimal stopping of discrete time parameter Markov chains with a finite state space containing at least one absorbing state a large number of results (*cf.* [1], [2]) are available. Note that (3.5) formulates the dynamic programming functional equations for the process.

Putting

$$\Phi_0 \overset{\text{def}}{=} C + \Psi - \Phi, \tag{3.6}$$

where $\Psi: U \to [0, \infty)$ is the unique solution of

$$\begin{aligned} k + \Delta_c \Psi &= 0, \qquad (x, y) \in U_i, \\ \Psi &= 0, \qquad (x, y) \in U_b, \end{aligned} \tag{3.7}$$

and

$$C \overset{\text{def}}{=} \max_{(x, y) \in U} (w - \Psi), \quad w_0 \overset{\text{def}}{=} C + \Psi - w \geqq 0, \tag{3.8}$$

* *Remark*: If $V \subset U$ and the process z_n started in $z_0 \in U - V$ is stopped as soon as it enters V then the entrance time into V is a stopping time.

then the relations (3.5) are equivalent with

$$\Phi_0 = \max\{w_0, \Delta_c \Phi_0 + \Phi_0\}, \qquad (x, y) \in U_i,$$
$$= C \qquad (x, y) \in U_b. \tag{3.9}$$

Here Φ_0 represents the optimal gain if no gain or costs are associated with transitions, whereas stopping at (x, y) yields a gain $w_0(x, y)$.

The system (3.9) is the standard form of optimal stopping processes as discussed in [1]. Applying the results obtained in [1] to (3.9), and interpreting them for (3.5) on behalf on the equivalence of (3.5) and (3.9), shows that for the optimal stopping process formulated by (3.5) an optimal policy exists for which the optimal cost function Φ is uniquely determined by (3.5) with its set T_s of stopping states given by

$$T_s = \{(x, y): (x, y) \in U, \ \Phi = w\}. \tag{3.10}$$

Further, the sequence Φ_n, $n = 1, 2, \ldots$, recursively defined by

$$\Phi_1 = w \qquad \text{for} \quad (x, y) \in U,$$
$$\Phi_{n+1} = \max\{w, k + \Delta_c \Phi_n + \Phi_n\}, \qquad n = 1, 2, \ldots; \ (x, y) \in U_i,$$
$$= 0, \qquad (x, y) \in U_b, \tag{3.11}$$

converges from above monotonically to Φ for every $(x, y) \in U$.

Moreover, the determination of the solution of (3.5) can also be formulated as a linear programming problem. This L.P. problem for (3.5) reads

$$\Phi = 0 \qquad \text{for} \quad (x, y) \in U_b,$$
$$\Delta_c \Phi + k \geqq 0, \quad \Phi \leqq w \quad \text{for} \quad (x, y) \in U_i, \tag{3.12}$$
$$\Sigma_{(x, y) \in U_i} \Phi \ \text{maximal}. \tag{3.13}$$

Obviously, the latter formulation describes the problem as a variational principle with constraints given by (3.12). Actually it shows that of all solutions of

$$\Phi = 0 \ \text{for} \ (x, y) \in U_b \quad \text{and} \quad \Delta_c \Phi + k \geqq 0 \ \text{for} \ (x, y) \in U_i$$

we need the maximal solution dominated by w.

The formulation (3.11) as well as (3.12), (3.13) are well suited for numerical evaluation of the solution.

From the results of the present and of the preceding section it is seen that the problem of the deflection of a hammock bounded by a roof w has the same mathematical formulation as the optimal stopping problem for a symmetric random walk discussed in this section.

4. Plastic-elastic torsion

If the hammock of Section 1 is replaced by a membrane stretched, with internal tension t per unit of length, over the rectangle and submitted to a constant upward pressure q per unit of surface then the relations which determine the deflection Φ in the equilibrium state are (*cf.* [5]).

$$\frac{q}{t} + \Delta\Phi = 0 \ \text{if} \ \Phi < w, \quad \text{for} \quad (x, y) \in S - B,$$

$$\frac{q}{t} + \Delta\Phi \geqq 0 \ \text{if} \ \Phi = w, \quad \text{for} \quad (x, y) \in S - B, \tag{4.1}$$

$$\Phi = 0, \tag{4.2}$$

and

$$\Phi, \ \frac{\partial\Phi}{\partial x} \ \text{and} \ \frac{\partial\Phi}{\partial y} \ \text{are continuous on} \ S, \tag{4.3}$$

with

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{4.4}$$

$$S \overset{\text{def}}{=} \{(x, y): 0 \leqq x \leqq a, 0 \leqq y \leqq b\}, \; S_i \overset{\text{def}}{=} \{(x, y): 0 < x < a, 0 < y < b\},$$

$$S_b \overset{\text{def}}{=} S - S_i,$$

B the boundary of the set $\{(x, y): \Phi = w\}$.

The relations hold for the case of small deflections of the membrane; (4.1) expresses the equilibrium conditions of an elementary part of the membrane; (4.2) represents the boundary condition, i.e. the membrane is fixed at the edges of the rectangle; the continuity of the derivatives of Φ in the equilibrium state stems from the fact that the only external forces on the membrane in the region S_i are pressures (line loads would lead to discontinuities in the derivatives).

The present formulation is the famous membrane-sandhill analogy of Nádai [3] for the plastic-elastic torsion of a prismatic bar with rectangular cross section. For the latter situation Φ represents the stress function from which the stress distribution in the cross section can be derived. In the domain where Φ touches the "roof", i.e. $\Phi = w$, plastic deformation occurs, in the complementary domain of the cross section the deformation is elastic.

The "roof" surface w (see Section 2), which is a surface of constant slope, can be physically realized by putting the maximum amount of dry sand on a horizontal plate of the same shape as the cross section of the bar. At every point of the surface so obtained its slope is the natural slope of the sand, it is determined by the friction between the sand grains, this slope is the constant α of Section 2. This property of a surface of constant slope and the similarity between the mathematical formulation of plastic-elastic torsion and that of small deflections of a membrane under pressure bounded by a surface of constant slope on its rim clarifies the name membrane-sandhill analogy.

Obviously, the conditions (4.1) are equivalent with

$$\begin{aligned} \Phi &= \min\left\{w, \frac{q}{t} d^2 + d^2 \Delta\Phi + \Phi\right\}, \quad \text{for} \quad (x, y) \in S - B, \\ &= 0 \qquad\qquad\qquad\qquad\qquad\quad \text{for} \quad (x, y) \in S_b, \\ &\Phi, \frac{\partial \Phi}{\partial x} \text{ and } \frac{\partial \Phi}{\partial y} \text{ continuous on } S; \end{aligned} \tag{4.5}$$

here d stands for the unit of length, it is introduced in (4.5) in order that all terms in (4.5) have the same dimension. It is readily seen that (2.4) is a finite difference approximation of (4.5).

5. Optimal stopping of Brownian motion

Let z_τ denote the position at time τ of a point subjected to Brownian motion without drift in S_i (*cf.* (4.4)), the points of the boundary S_b are assumed to be absorbing; D shall stand for the diffusion coefficient. As in Section 3 we shall consider stopping times and associated stopping processes. Per unit of time that the point moves a cost $h > 0$ is incurred, whereas stopping at (x, y) involves a cost $w(x, y)$, w being the "roof" function of Section 2. For a given stopping process with stopping time σ and starting at (x, y) the expected costs are given by

$$\phi(x, y) \overset{\text{def}}{=} E\{h\sigma + w(z_\sigma)|z_0 = (x, y)\}, \qquad (x, y) \in S. \tag{5.1}$$

Before discussing optimal stopping for the process z_τ we shall consider the costs of sojourn in a circle C with boundary Γ, radius R and center at (x, y), when the process starts at $(\xi, \eta) \in C$ and is stopped as soon as the moving point reaches Γ. Let $\tau(\xi, \eta)$ denote the first entrance time from (ξ, η) into Γ, then (*cf.* [2])

$$E\{\tau(\xi,\eta)\} = \frac{1}{2D}\{R^2-(x-\xi)^2-(y-\eta)^2\}, \qquad (\xi,\eta)\in C\;; \tag{5.2}$$

and

$$\Psi(\xi,\eta) \stackrel{\text{def}}{=} hE\{\tau(\xi,\eta)\} \tag{5.3}$$

are the expected costs of the sojourn in C when starting at (ξ, η). Consequently, $hR^2/2D$ may be considered as the expected costs of the sojourn time in C when starting at the center of a circle with radius R and absorbing boundary.

Obviously Ψ and its first derivatives are continuous on C. However, if we consider the situation that an extra cost h_1 is incurred every time that the moving point passes the boundary Γ_1 of a circle C_1 with radius $\frac{1}{2}R$ and center at (x, y), then the expected costs of the sojourn in C, i.e. until the first entrance into Γ, is still a continuous function of the starting point (ξ, η), but its derivatives are discontinuous at the points of Γ_1; however, such costs (as h_1) will not be considered in our model.

The problem of optimal stopping of the process described above with $h=0$ has been investigated in [2] and in [6]. From the results of [6] applied to our situation it can be shown that an optimal stopping time σ_0 exists with cost function

$$\Phi \stackrel{\text{def}}{=} \inf_{\sigma} \phi\,. \tag{5.4}$$

Defining

$$T_s \stackrel{\text{def}}{=} \{(x,y) : \Phi = w\}, \quad T_c \stackrel{\text{def}}{=} \{(x,y) : \Phi < w\}, \tag{5.5}$$

then the optimal policy is to stop at the points of T_s, and to continue at points of T_c until the first entrance into T_s. Since Brownian motion paths are continuous with probability one, Φ will be continuous (*cf.* [6]), so that since the "roof" function w is also continuous, the set T_s is closed; denote by B its boundary.

For $(x, y)\in T_c$ denote by C a circle with center at (x, y), boundary Γ and such that $C\bigcup\Gamma\subset T_c$. With $\tau(x, y)$ the first entrance time from (x, y) into Γ and $z_{\tau(x,y)}$ the hitting point of Γ, so that on behalf of symmetry of the Brownian motion $z_{\tau(x,y)}$ is uniformly distributed on Γ, it follows since stopping at (x, y) is more costly then moving from (x, y) that (*cf.* (5.2))

$$\begin{aligned}\Phi(x,y) &= \frac{1}{2\pi R}\int_\Gamma \Phi(z_{\tau(x,y)})\,d\gamma + hE\{\tau(x,y)\}\\ &= \frac{1}{2\pi R}\int_\Gamma \Phi(z_{\tau(x,y)})\,d\gamma + \frac{h}{\pi D}\int_0^{2\pi}\left\{\int_0^R r\log\frac{R}{r}\,dr\right\}d\beta\,.\end{aligned} \tag{5.6}$$

From the continuity of Φ, from (5.6), and from a well-known theorem of potential theory it is readily seen that Φ is regular on C and that

$$\tfrac{1}{2}D\Delta\Phi + h = 0 \quad \text{for} \quad (x,y)\in T_c\,. \tag{5.7}$$

Suppose next that $(x, y)\in T_s$ and (x, y) not a point of the boundary B of T_s, so that a circle C with boundary Γ, radius R and center at (x, y) exists such that $C\bigcup\Gamma\subset T_s$, hence

$$\Phi = w \text{ for all } (\xi,\eta)\in C\bigcup\Gamma\,.$$

Since moving from (x, y) is at least as costly as stopping at (x, y) we have

$$w(x,y) \leqq hE\{\tau(x,y)\} + E\{w(z_{\tau(x,y)})\}$$

or

$$h + [E\{w(z_{\tau(x,y)})\} - w(x,y)]/E\{\tau(x,y)\} \geqq 0\,. \tag{5.8}$$

The "roof" function is a continuous function of constant slope, i.e.

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \alpha^2 = \text{constant on } S\,, \tag{5.9}$$

at all points where the derivatives exist. Obviously, at these points the second derivatives exist also, and hence on behalf of a result in [2], the left-hand side of (5.8) has a limit for $R \to 0$. Proceeding to the limit it follows (*cf.* [2]) that

$$\tfrac{1}{2} D \Delta w + h \geqq 0 \text{ or equivalently } \tfrac{1}{2} D \Delta \Phi + h \geqq 0 \tag{5.10}$$

for (x, y) an internal point of T_s at which the derivatives of w exist.

At the points of the boundary B of Γ_s where $\phi = w$ the derivatives $\partial\phi/\partial x$, $\partial\phi/\partial y$ should be continuous* (*cf.* the remarks above about the continuity of ψ), so that

$$\frac{\partial \phi}{\partial x} = \frac{\partial w}{\partial x}, \quad \frac{\partial \phi}{\partial y} = \frac{\partial w}{\partial y} \text{ for } (x, y) \in B, \tag{5.11}$$

(*cf.* also [6] for the derivation of the conditions (5.11)).

The fact that the derivatives of Φ should be continuous for all $(x, y) \in S_i$ implies on behalf of (5.5) that T_s does not contain points at which the derivatives of w do not exist.

Consequently, we have from (5.7) and (5.10)

$$\begin{aligned} &\Phi = \min\left\{ w, \frac{2d^2 h}{D} + d^2 \Delta \Phi + \Phi \right\} && \text{for} \quad (x, y) \in S - B \\ &\Phi = 0 && \text{for} \quad (x, y) \in S_b, \\ &\Phi, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \text{ are continuous on } S\,; \end{aligned} \tag{5.12}$$

here d stands for the unit of length, it is introduced in (5.12) in order that all quantities in (5.12) have the same dimension.

Obviously, the relations (3.5) are the finite difference approximations of the relations (5.12). Moreover, the first relation of (5.12) may be considered as the dynamical programming functional equation of the stopping problem, since on behalf of (5.6) and (5.8) we have

$$\Phi = \min\left\{ w, 2\frac{R^2 h}{D} + R^2 \Delta \Phi + \Phi + o(R^2) \right\} \text{ for } R \to 0\,.$$

6. Concluding remarks

The discussions of the preceding sections show clearly that the free boundary problems encountered in plastic-elastic torsion of cylindrical bars, in the deflection of membranes bounded by a roof and in optimal stopping of Brownian motion are all of a similar type. Although the discussion has been restricted to a simple domain S, i.e. the rectangular, for more complicated domains the discussion will not be essentially different as long as its boundary is sufficiently regular. The derivation of the mathematical formulation of the plastic-elastic torsion as well as that of the membrane-sandhill analogy have been omitted, they may be found in [3] and [4]. The derivations of optimal stopping of Brownian motion as given in section 5 contain several heuristic arguments; for a more rigorous derivation the techniques of [6] should be applied, *cf.* also [2]. It should be emphasized that the membrane analogy is extremely helpful in obtaining a good insight in optimal stopping of Brownian motion.

Finally, in continuum mechanics and heat transfer several interesting free boundary problems for the Laplace and the Poisson equation occur, see e.g. [7]; it is certainly of some interest to investigate whether these problems can be formulated in the context of optimal stopping of Brownian motion.

For further and more detailed information of the relations between free boundary problems and optimal stopping see [8] and [9].

* The cost (roof) function is concave. For a convex cost function with for instance a sharp edge the derivatives need not be continuous.

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